

# THE LATTICE SCHWARZIAN KDV EQUATION AND ITS SYMMETRIES

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**ABSTRACT.** In this paper we present a set of results on the symmetries of the lattice Schwarzian Korteweg-de Vries (ISKdV) equation. We construct the Lie point symmetries and, using its associated spectral problem, an infinite sequence of generalized symmetries and master symmetries. We finally show that we can use master symmetries of the ISKdV equation to construct non-autonomous non-integrable generalized symmetries.

## 1. INTRODUCTION

The lattice version of the Schwarzian Korteweg-de Vries (ISKdV) equation

$$(1) \quad w_t = w_{xxx} - \frac{3w_{xx}^2}{2w_x},$$

is given by the nonlinear partial difference equation [18, 20]:

$$(2) \quad \mathbb{Q} \doteq \alpha_1 (x_{n,m} - x_{n,m+1}) (x_{n+1,m} - x_{n+1,m+1}) - \\ - \alpha_2 (x_{n,m} - x_{n+1,m}) (x_{n,m+1} - x_{n+1,m+1}) = 0.$$

Eq. (2) involves just four lattice points which lay on two orthogonal infinite lattices and are situated at the vertices of an elementary square. It is a lattice equation on *quad-graphs* belonging to the recent classification presented in [3], where the *3D consistency* is used as a tool to establish its integrability. It is a subcase of the first element of the *Q-list*, namely the equation  $\mathbb{Q}_1$ , with  $\delta = 0$ .

As far as we know the ISKdV equation has been introduced for the first time by Nijhoff, Quispel and Capel [20] in 1983. A review of results about the ISKdV equation can be found in [16, 18].

Since we have two discrete independent variables, i.e.  $n$  and  $m$ , we can perform the continuous limit in two steps. Each step is achieved by shrinking the corresponding lattice step to zero and sending to infinity the number of points of the lattice.

In the first step, setting  $\alpha_1 \doteq q^2$ ,  $\alpha_2 \doteq p^2$ , we define  $x_{n,m} \doteq \tilde{x}_k(\tau)$ , where  $k \doteq n + m$  and  $\tau \doteq \delta m$ , being  $\delta \doteq p - q$ . Considering the limits  $m \rightarrow \infty$ ,  $\delta \rightarrow 0$ , we get the differential-difference equation

$$(3) \quad \frac{d\tilde{x}_k(\tau)}{d\tau} = \frac{2(\tilde{x}_{k+1} - \tilde{x}_k)(\tilde{x}_{k-1} - \tilde{x}_k)}{p(\tilde{x}_{k-1} - \tilde{x}_{k+1})}.$$

The second continuous limit of eq. (2) is performed by taking in eq. (3)  $\tilde{x}_k(\tau) \doteq w(x, t)$ , with  $x \doteq 2(k + \tau/p)/p$  and  $t \doteq 2(k/3 + \tau/p)/p^3$ . If we carry out the limit  $p \rightarrow \infty, k \rightarrow \infty, \tau \rightarrow \infty$ , in such a way that  $x$  and  $t$  remain finite, then eq. (3) is transformed into the continuous Schwarzian KdV equation (1).

Integrable equations possess an infinite set of symmetries. Few of them are *point symmetries*, i.e. symmetries whose infinitesimal generators depend just on the independent and dependent variables, while an infinite denumerable number of them are *generalized symmetries*. These latter ones depend also on the derivatives of the dependent variable, for the continuous independent variables, and on a few lattice points, if the independent variables are discrete. The presence of this infinite Lie algebra of symmetries is one of the most important features of the integrability of a given nonlinear equation and it has been used with profit in the past to provide integrability tests for several partial differential equations in  $\mathbb{R}^2$  and differential-difference equations[14, 22].

To be able to introduce an integrability test based on symmetries, one needs to understand the structure of the infinite dimensional symmetry algebra of the integrable equations. In the case of completely discrete equations the situation is not as clear as for the differential-difference or the partial-differential case. Results in this direction have been obtained some time ago for the discrete-time Toda lattice [11] and recently for the lattice potential Korteweg-de Vries equation [12]. However, the Toda Lattice and the lattice potential KdV equation are just examples and more examples are needed to get a sufficiently general idea of the possible structures which may appear.

The present paper is part of this research and is devoted to the study of the LSKdV equation exactly from this point of view. In section 2 we present some old and new results on the integrability of the LSKdV equation. Section 3 is devoted to the construction of its Lie point symmetries while in section 4 we consider its generalized symmetries. Finally, section 5 contains some concluding remarks.

## 2. THE INTEGRABILITY OF THE LSKdV EQUATION

Eq. (2) has been obtained firstly by the direct linearization method [20]. In [18] one can find its associated spectral problem, which, as this equation is part of the Adler-Bobenko-Suris classification, can be obtained using a well-defined procedure [4, 5, 17].

Its Lax pair is given by the following overdetermined system of matrix equations for the vector  $\Psi_{n,m}^\lambda \doteq (\psi_{n,m}^1(\lambda), \psi_{n,m}^2(\lambda))^T$ ,  $\lambda \in \mathbb{R}$ ,

$$(4a) \quad \Psi_{n+1,m}^\lambda = L_{n,m}^\lambda \Psi_{n,m}^\lambda,$$

$$(4b) \quad \Psi_{n,m+1}^\lambda = M_{n,m}^\lambda \Psi_{n,m}^\lambda,$$

where

$$L_{n,m}^\lambda \doteq \begin{pmatrix} 1 & x_{n,m} - x_{n+1,m} \\ \lambda \alpha_1 (x_{n,m} - x_{n+1,m})^{-1} & 1 \end{pmatrix},$$

and

$$M_{n,m}^\lambda \doteq \begin{pmatrix} 1 & x_{n,m} - x_{n,m+1} \\ \lambda \alpha_2 (x_{n,m} - x_{n,m+1})^{-1} & 1 \end{pmatrix}.$$

The consistency of eqs. (4) implies the discrete Lax equation

$$L_{n,m+1}^\lambda M_{n,m}^\lambda = M_{n+1,m}^\lambda L_{n,m}^\lambda.$$

We can rewrite eqs. (4) in scalar form in terms of  $\psi_{n,m} \doteq \psi_{n,m}^1(\lambda)$ :

$$(5a) \quad \begin{aligned} & (x_{n,m} - x_{n+1,m}) \psi_{n+2,m} + (x_{n+2,m} - x_{n,m}) \psi_{n+1,m} + \\ & + (1 - \lambda \alpha_1) (x_{n+1,m} - x_{n+2,m}) \psi_{n,m} = 0, \end{aligned}$$

$$(5b) \quad \begin{aligned} & (x_{n,m} - x_{n,m+1}) \psi_{n,m+2} + (x_{n,m+2} - x_{n,m}) \psi_{n,m+1} + \\ & + (1 - \lambda \alpha_2) (x_{n,m+1} - x_{n,m+2}) \psi_{n,m} = 0. \end{aligned}$$

It is worthwhile to observe here that eq. (2) and the Lax equations (5) are invariant under the discrete symmetry obtained by interchanging at the same time  $n$  with  $m$  and  $\alpha_1$  with  $\alpha_2$ .

To get meaningful Lax equations the field  $x_{n,m}$  cannot go asymptotically to a constant but must be written as  $x_{n,m} \doteq u_{n,m} + \beta_0 m + \alpha_0 n$ , where  $\alpha_0$  and  $\beta_0$  are constants related to  $\alpha_1$  and  $\alpha_2$  by the condition  $\alpha_1 \beta_0^2 = \alpha_2 \alpha_0^2$ . Under this transformation of the dependent variable, the lSKdV equation and its Lax pair can be respectively rewritten as

$$(6) \quad \begin{aligned} & \alpha_1 (u_{n,m} - u_{n,m+1} - \beta_0) (u_{n+1,m} - u_{n+1,m+1} - \beta_0) = \\ & = \alpha_2 (u_{n,m} - u_{n+1,m} - \alpha_0) (u_{n,m+1} - u_{n+1,m+1} - \alpha_0), \end{aligned}$$

and

$$(7a) \quad (1 + v_{n,m}^{(n)}) \psi_{n+2,m} - (2 + v_{n,m}^{(n)}) \psi_{n+1,m} + (1 - \lambda \alpha_1) \psi_{n,m} = 0,$$

$$(7b) \quad (1 + v_{n,m}^{(m)}) \psi_{n,m+2} - (2 + v_{n,m}^{(m)}) \psi_{n,m+1} + (1 - \lambda \alpha_2) \psi_{n,m} = 0,$$

where

$$v_{n,m}^{(n)} \doteq \frac{u_{n+2,m} - 2u_{n+1,m} + u_{n,m}}{u_{n+1,m} - u_{n,m} + \alpha_0}, \quad v_{n,m}^{(m)} \doteq \frac{u_{n,m+2} - 2u_{n,m+1} + u_{n,m}}{u_{n,m+1} - u_{n,m} + \beta_0},$$

where  $u_{n,m} \rightarrow c$ , with  $c \in \mathbb{R}$ , as  $n$  and  $m$  go to infinity.

One can construct the class of differential-difference equations associated with eq. (7a) by requiring the existence of a set of operators  $M_n$  such that [7]

$$L_n \psi_n = \mu \psi_n, \quad \frac{d\psi_n}{dt} = -M_n \psi_n, \quad \frac{dL_n}{dt} = [L_n, M_n].$$

with  $L_n \doteq (1 + v_{n,m}^{(n)}(t)) E^2 - (2 + v_{n,m}^{(n)}(t)) E$ . Here  $E$  is the (positive) shift operator in the variable  $n$ . If  $d\mu/dt = 0$  the class of differential-difference

equations one so obtains will be called *isospectral*, while if  $d\mu/dt \neq 0$  it will be called *non-isospectral*.

As the field  $v_{n,m}^{(n)}(t)$  appears multiplying both  $E$  and  $E^2$  the expression of the recursive operator turns out to be extremely complicate, containing triple sums and products of the dependent fields. So we look for transformations of the spectral problem (7a) which reduce it to a simpler form in which the potential will appear just once. There are two different discrete spectral problems involving three lattice points, the discrete Schrödinger spectral problem introduced by Case [8],

$$(8) \quad \phi_{n-1} + a_n \phi_{n+1} + b_n \phi_n = \lambda \phi_n,$$

which is associated with the Toda and Volterra differential-difference equations [10] and the asymmetric discrete Schrödinger spectral problem introduced by Shabat and Boiti et al. [6, 21],

$$(9) \quad \phi_{n+2} = \frac{2p}{s_n} \phi_{n+1} + \lambda \phi_n.$$

The latter one has been used to solve the so called discrete KdV equation [15]. In eqs. (8–9) the functions  $a_n, b_n, s_n$  may depend parametrically on a *continuous variable*  $t$  but also on a *discrete variable*  $m$ . As all three spectral problems (7a–8–9) involve just three points on the lattice we can relate them by a gauge transformation  $\psi_n \doteq f_n(\mu) g_n(\{u_{n,m}\}) \phi_n$ , where  $\{u_{n,m}\} \doteq (u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots)$ . These transformations give rise to a Miura transformation between the involved fields. For instance, when we transform the spectral problem (7a) into the discrete Schrödinger spectral problem (8) we get

$$(10a) \quad b_n \doteq b_{n,m} = 0,$$

$$(10b) \quad a_n \doteq a_{n,m} = \frac{4(u_{n+1,m} - u_{n,m} + \alpha_0)^2}{(u_{n+2,m} - u_{n,m} + 2\alpha_0)(u_{n+1,m} - u_{n-1,m} + 2\alpha_0)}.$$

If we transform the spectral problem given in eq. (7a) into the asymmetric discrete Schrödinger spectral problem (9) the relation between the fields  $s_n \doteq s_{n,m}$  and  $u_{n,m}$  is more involved as it is expressed in terms of infinite products.

We will use in section 4 the transformation (10b) and the equivalent one between (7b) and (8) which will define a field  $\tilde{a}_m$  given by eq. (10b) with  $n$  and  $m$  and  $\alpha_0$  and  $\beta_0$  interchanged, to build the generalized symmetries of eq. (6) from the nonlinear differential-difference equations associated with the spectral problem (8) [10] with  $b_n$  given by eq. (10a).

### 3. POINT SYMMETRIES OF THE LSKdV EQUATION

Here we look for the Lie point symmetries of eq. (2), with  $\alpha_1 \neq \alpha_2$ , using the technique introduced in [13]. The Lie point symmetries we obtain in this way turn out to be the same as those for (6).

The Lie symmetries of the lSKdV equation (2) are given by those continuous transformations which leave the equation invariant. From the infinitesimal point of view they are obtained by requiring the infinitesimal invariant condition

$$(11) \quad \text{pr } \hat{X}_{n,m} \mathbb{Q} \Big|_{\mathbb{Q}=0} = 0,$$

where, as we keep the form of the lattice invariant,

$$(12) \quad \hat{X}_{n,m} = \Phi_{n,m}(x_{n,m}) \partial_{x_{n,m}}.$$

By  $\text{pr } \hat{X}_{n,m}$  we mean the prolongation of the infinitesimal generator  $\hat{X}_{n,m}$  to the other three points appearing in  $\mathbb{Q} = 0$ , i.e.  $x_{n+1,m}$ ,  $x_{n,m+1}$  and  $x_{n+1,m+1}$ .

Solving the equation  $\mathbb{Q} = 0$  w.r.t.  $x_{n+1,m+1}$  and substituting it in eq. (11) we get a functional equation for  $\Phi_{n,m}(x_{n,m})$ . Looking at its solutions in the form  $\Phi_{n,m}(x_{n,m}) = \sum_{k=0}^{\gamma} \Phi_{n,m}^{(k)} x_{n,m}^k$ ,  $\gamma \in \mathbb{N}$ , we see that in order to balance the leading order in  $x_{n,m}$ , if  $\alpha_1 \neq \alpha_2$ ,  $\gamma$  cannot be greater than 2, and thus must belong to the interval  $[0, 2]$ . Equating now to zero the coefficients of the powers of  $x_{n,m}$ ,  $x_{n+1,m}$  and  $x_{n,m+1}$  we get an overdetermined system of determining equations. Solving the resulting difference equations we find that the functions  $\Phi_{n,m}^{(i)}$ 's,  $i = 0, 1, 2$ , must be constants. Hence the infinitesimal generators of the algebra of Lie point symmetries are given by

$$\hat{X}_{n,m}^{(0)} = \partial_{x_{n,m}}, \quad \hat{X}_{n,m}^{(1)} = x_{n,m} \partial_{x_{n,m}}, \quad \hat{X}_{n,m}^{(2)} = x_{n,m}^2 \partial_{x_{n,m}}.$$

The generators  $\hat{X}_{n,m}^{(i)}$ ,  $i = 0, 1, 2$ , span the Lie algebra  $\mathfrak{sl}(2)$ :

$$\begin{aligned} [\hat{X}_{n,m}^{(0)}, \hat{X}_{n,m}^{(1)}] &= \hat{X}_{n,m}^{(0)}, & [\hat{X}_{n,m}^{(1)}, \hat{X}_{n,m}^{(2)}] &= \hat{X}_{n,m}^{(2)}, \\ [\hat{X}_{n,m}^{(0)}, \hat{X}_{n,m}^{(2)}] &= 2 \hat{X}_{n,m}^{(1)}. \end{aligned}$$

We can write down the group transformation by integrating the differential-difference equation

$$(13) \quad \frac{d\tilde{x}_{n,m}(\epsilon)}{d\epsilon} = \Phi_{n,m}(\tilde{x}_{n,m}(\epsilon)),$$

with the initial condition  $\tilde{x}_{n,m}(\epsilon = 0) = x_{n,m}$ . We get the Möbius transformation [4, 5, 18]

$$\tilde{x}_{n,m}(\epsilon_0, \epsilon_1, \epsilon_2) = \frac{(\epsilon_0 + x_{n,m}) e^{\epsilon_1}}{1 - \epsilon_2 (\epsilon_0 + x_{n,m}) e^{\epsilon_1}},$$

where the  $\epsilon_i$ 's are the parameters associated with the infinitesimal generators  $\hat{X}^{(i)}$ ,  $i = 0, 1, 2$ .

We finally notice that, in the case when  $\alpha_1 = \alpha_2$  eq. (2) simplifies to the product of two linear discrete wave equations:

$$(x_{n,m} - x_{n+1,m+1})(x_{n+1,m} - x_{n,m+1}) = 0,$$

which is trivially solved by taking  $x_{n,m} = f_{n \pm m}$  and the Lie point symmetries belong to an infinite dimensional Lie algebra.

#### 4. GENERALIZED SYMMETRIES OF THE LSKdV EQUATION

A generalized symmetry is obtained when the function  $\Phi_{n,m}$  appearing in eq. (12) depends on  $\{x_{n,m}\}$  and not only on  $x_{n,m}$ . A way to obtain it, is to look at those differential-difference equations (13) associated with the spectral problem (8) which are compatible with eq. (2). From eqs. (10) we see that the LSKdV equation can be associated with the discrete Schrödinger spectral problem when  $b_{n,m} = 0$ , i.e. when the associated hierarchy of differential-difference equations is given by the Volterra hierarchy [10]. So, applying the Miura transformation (10) to the differential-difference equations of the Volterra hierarchy we can obtain the symmetries of the LSKdV equation. The Miura transformation (10) preserves the integrability of the Volterra hierarchy if  $u_{n,m} \rightarrow c$ , with  $c \in \mathbb{R}$ , as  $n$  and  $m$  go to infinity.

The procedure to get the generalized symmetries for the LSKdV is better shown on a specific example, the case of the Volterra equation, an isospectral deformation of the spectral problem (8):

$$(14) \quad \frac{da_{n,m}(\epsilon_0)}{d\epsilon_0} = a_{n,m}(a_{n+1,m} - a_{n-1,m}).$$

Let us substitute the Miura transformation, given by eq. (10b), into eq. (14) and let us assume that

$$\frac{du_{n,m}(\epsilon_0)}{d\epsilon_0} = F_{n,m}(u_{n-1,m}, u_{n,m}, u_{n+1,m}).$$

Eq. (14) is thus a functional equation for  $F_{n,m}$  which can be solved as we did in the previous section by comparing powers at infinity or by transforming it into an overdetermined system of linear partial differential equations [1, 2]. In this way we get, up to a point transformation,

$$(15) \quad F_{n,m} = \frac{4(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0}.$$

Eq. (15) is nothing else but eq. (3). One can verify that eq. (15) is a generalized symmetry of the LSKdV equation (6) by proving that  $\Phi_{n,m} = F_{n,m}(u_{n-1,m}, u_{n,m}, u_{n+1,m})$  satisfies eq. (11).

If we start from a higher equation of the isospectral Volterra hierarchy

$$\begin{aligned} \frac{da_{n,m}(\epsilon_1)}{d\epsilon_1} = & a_{n,m} [a_{n-1,m}(a_{n-2,m} + a_{n-1,m} + a_{n,m} - 4) - \\ & - a_{n+1,m}(a_{n+2,m} + a_{n+1,m} + a_{n,m} - 4)], \end{aligned}$$

we get a second generalized symmetry of the lSKdV equation requiring  $F_{n,m} = F_{n,m}(u_{n-2,m}, u_{n-1,m}, u_{n,m}, u_{n+1,m}, u_{n+2,m})$ . It reads

$$(16) \quad F_{n,m} = \frac{(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{(u_{n+1,m} - u_{n-1,m} + 2\alpha_0)^2} \times \\ \times \left[ \frac{(u_{n+2,m} - u_{n+1,m} + \alpha_0)(u_{n-1,m} - u_{n,m} - \alpha_0)}{u_{n+2,m} - u_{n,m} + 2\alpha_0} - \frac{(u_{n-1,m} - u_{n-2,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n,m} - u_{n-2,m} + 2\alpha_0} \right].$$

This procedure could be clearly carried out for any equation of the Volterra hierarchy [10] giving a hierarchy of symmetries for the lSKdV equation.

If we consider the non-isospectral hierarchy the only local equation is [10]

$$\frac{da_{n,m}(\sigma)}{d\sigma} = a_{n,m} [a_{n,m} - (n-1)a_{n-1,m} + (n+2)a_{n+1,m} - 4]$$

and it provides, up to a Lie point symmetry, two local equations:

$$(17) \quad \frac{du_{n,m}(\sigma_0)}{d\sigma_0} = u_{n,m} + \alpha_0 n,$$

$$(18) \quad \frac{du_{n,m}(\sigma_1)}{d\sigma_1} = \frac{4n(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0}.$$

Eq. (17) is obtained as the multiplicative factor of an integration constant. One can easily show that eq. (17) is not a symmetry of the lSKdV equation but it commutes with all its known symmetries. Eq. (18) is a master symmetry [9]: it does not commute with the lSKdV equation but commuting it with eq. (15) one gets eq. (16) and commuting it with eq. (16) one gets a higher order symmetry. So through it one can construct a hierarchy of generalized symmetries of the lSKdV equation.

In the construction of generalized symmetries for the differential-difference Volterra equation [10] one was able to construct a symmetry from the master symmetry (18) by combining it with the second isospectral symmetry (16) multiplied by  $t$ . This seems not to be the case for difference-difference equations. As was showed in [12] for the case of the lattice potential KdV equation, there is no combination of eq. (18) with isospectral symmetries which gives us a symmetry of eq. (6).

As the lSKdV equation admits a discrete symmetry corresponding to an exchange of  $n$  with  $m$  and  $\alpha_1$  with  $\alpha_2$ , one can construct another class of generalized and master symmetries by considering the equations obtained from the spectral problem (8) in the  $m$  lattice variable depending on the

potential  $\tilde{a}_m$ . In this way we get:

$$\begin{aligned}\frac{du_{n,m}(\tilde{\epsilon}_0)}{d\tilde{\epsilon}_0} &= \frac{4(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0}, \\ \frac{du_{n,m}(\tilde{\epsilon}_1)}{d\tilde{\epsilon}_1} &= \frac{(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{(u_{n,m+1} - u_{n,m-1} + 2\beta_0)^2} \times \\ &\quad \times \left[ \frac{(u_{n,m+2} - u_{n,m+1} + \beta_0)(u_{n,m-1} - u_{n,m} - \beta_0)}{u_{n,m+2} - u_{n,m} + 2\beta_0} - \right. \\ &\quad \left. - \frac{(u_{n,m-1} - u_{n,m-2} + \beta_0)(u_{n,m} - u_{n,m+1} - \alpha_0)}{u_{n,m} - u_{n,m-2} + 2\beta_0} \right].\end{aligned}$$

and

$$\begin{aligned}\frac{du_{n,m}(\tilde{\sigma}_0)}{d\tilde{\sigma}_0} &= u_{n,m} + \beta_0 m, \\ \frac{du_{n,m}(\tilde{\sigma}_1)}{d\tilde{\sigma}_1} &= \frac{4m(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0}.\end{aligned}$$

A different class of symmetries can be obtained applying the following theorem, introduced in [12], which provides a constructive tool to obtain generalized symmetries for the LSKdV equation (2).

**Theorem 1.** *Let  $\mathbb{Q}(u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots; \alpha_1, \alpha_2) = 0$  be an integrable partial difference equation invariant under the discrete symmetry  $n \leftrightarrow m$ ,  $\alpha_1 \leftrightarrow \alpha_2$ . Let  $\hat{Z}_n$  be the differential operator*

$$\hat{Z}_n \doteq Z_n(u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots; \alpha_1, \alpha_2) \partial_{u_{n,m}},$$

such that

$$\text{pr } \hat{Z}_n \mathbb{Q} \Big|_{\mathbb{Q}=0} = a g_{n,m}(u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots; \alpha_1, \alpha_2),$$

where  $g_{n,m}(u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots; \alpha_1, \alpha_2)$  is invariant under the discrete symmetry  $n \leftrightarrow m$ ,  $\alpha_1 \leftrightarrow \alpha_2$  and  $a$  is an arbitrary constant. Then

$$\left( \frac{1}{a} \text{pr } \hat{Z}_n - \frac{1}{b} \text{pr } \hat{Z}_m \right) \mathbb{Q} \Big|_{\mathbb{Q}=0} = 0,$$

where the operator  $\hat{Z}_m \doteq Z_m(u_{n,m}, u_{n,m\pm 1}, u_{n\pm 1,m}, \dots; \alpha_2, \alpha_1) \partial_{u_{n,m}}$  is obtained from  $\hat{Z}_n$  under  $n \leftrightarrow m$ ,  $\alpha_1 \leftrightarrow \alpha_2$ , so that

$$\text{pr } \hat{Z}_m \mathbb{Q} \Big|_{\mathbb{Q}=0} = b g_{n,m}(u_{n,m}, u_{n\pm 1,m}, u_{n,m\pm 1}, \dots; \alpha_2, \alpha_1),$$

being  $b$  a constant. So

$$\hat{Z}_{n,m} \doteq \frac{1}{a} \hat{Z}_n - \frac{1}{b} \hat{Z}_m$$

is a symmetry of  $\mathbb{Q} = 0$ .



Using this theorem it is easy to show that from the master symmetry (18) we can construct a generalized symmetry, given by

$$\begin{aligned} \frac{du_{n,m}(\epsilon)}{d\epsilon} = & \frac{4n(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0} + \\ & + \frac{4m(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0}. \end{aligned}$$

The above symmetry has been implicitly used, together with point symmetries, by Nijhoff and Papageorgiou [19] to perform the similarity reduction of the lSKdV equation and get a discrete analogue of the Painlevé II equation.

## 5. CONCLUSIONS

In this paper we have constructed by group theoretical methods the symmetries of the lSKdV equation. The Lie point symmetry algebra provides a Möbius transformation of the dependent variable. The generalized symmetries are obtained in a constructive way by considering the spectral problem associated with the lSKdV equation. As was shown in [16, 18] we can associate with this integrable lattice equation an asymmetric discrete Schrödinger spectral problem with two potentials. Since it is not easy to construct its associated evolution equations we have transformed the obtained spectral problem into the standard discrete Schrödinger spectral problem, whose corresponding evolution equations are the Toda and Volterra hierarchies. In this way we have constructed two classes of integrable symmetries, associated with the  $n$  and  $m$  components of the Lax pair and we have obtained symmetries starting from the master symmetries. So we have obtained in a group theoretical framework the nonlinear reductions considered in [19]

We left to future work to prove if the two classes of isospectral symmetries are independent and whether the symmetries obtained from the master symmetries are integrable or not. Another open problem is the construction of the recursive operator for the lSKdV symmetries.

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